Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 1

Thursday 19 May 2016 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

2216 – 7101

8 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

The set P contains all prime numbers less than 2500.
The set Q is the set of all subsets of P.

(a) Explain why only one of the following statements is true

(i) $17 \subset P$;

(ii) $\{7, 17, 37, 47, 57\} \in Q$;

(iii) $\phi \subset Q$ and $\phi \in Q$, where ϕ is the empty set. [4]

The set S contains all positive integers less than 2500.
The function $f : S \to Q$ is defined by $f(s)$ as the set of primes exactly dividing s, for $s \in S$.
For example $f(4) = \{2\}$, $f(45) = \{3, 5\}$.

(b) (i) State the value of $f(1)$, giving a reason for your answer.

(ii) Find $n\left(f(2310)\right)$. [4]

(c) Determine whether or not f is

(i) injective;

(ii) surjective. [4]

2. [Maximum mark: 11]

The lifetime, in years, of a randomly chosen basic vacuum cleaner is assumed to be modelled by the normal distribution $B \sim \mathrm{N}(14\,,\,3^2)$.

(a) Find $\mathrm{P}\left(B > \mathrm{E}(B) + \frac{1}{2}\sqrt{\mathrm{Var}(B)}\right)$. [2]

(b) Find the probability that the total lifetime of 7 randomly chosen basic vacuum cleaners is less than 100 years. [4]

The lifetime, in years, of a randomly chosen robust vacuum cleaner is assumed to be modelled by the normal distribution $R \sim \mathrm{N}(20\,,\,4^2)$.

(c) Find the probability that the total lifetime of 5 randomly chosen robust vacuum cleaners is greater than the total lifetime of 7 randomly chosen basic vacuum cleaners. [5]

3. [Maximum mark: 8]

Consider the Diophantine equation $7x - 5y = 1\,,\ x\,,\ y \in \mathbb{Z}$.

(a) Find the general solution to this equation. [3]

(b) Hence find the solution with minimum positive value of xy. [2]

(c) Find the solution satisfying $xy = 2014$. [3]

4. [Maximum mark: 10]

All members of a large athletics club take part in an annual shotput competition.
The following data give the distances achieved, in metres, by a random selection of 10 members of the club in the 2016 competition

$$11.8, 14.3, 13.8, 10.3, 14.9, 14.7, 12.4, 13.9, 14.0, 11.7$$

The president of the club wishes to test whether these data provide evidence that distances achieved have increased since the 2015 competition, when the mean result for the club was $12.4\,\text{m}$. You may assume that the distances achieved follow a normal distribution with mean μ, variance σ^2, and that the membership of the club has not changed from 2015 to 2016.

(a) State suitable hypotheses. [1]

(b) (i) Give a reason why a t test is appropriate and write down its degrees of freedom.

(ii) Find the critical region for testing at each of the 5% and 10% significance levels. [4]

(c) (i) Find unbiased estimates of μ and σ^2.

(ii) Find the value of the test statistic. [3]

(d) State the conclusions that the president of the club should reach from this test, giving reasons for your answer. [2]

5. [Maximum mark: 8]

Consider the curve C given by $y = x^3$.
The tangent at a point P on C meets the curve again at Q. The tangent at Q meets the curve again at R. Denote the x-coordinates of P, Q and R, by x_1, x_2 and x_3 respectively where $x_1 \neq 0$. Show that, x_1, x_2, x_3 form the first three elements of a divergent geometric sequence.

6. [Maximum mark: 8]

Consider the recurrence relation $H_{n+1} = 2H_n + 1$, $n \in \mathbb{Z}^+$ where $H_1 = 1$.

(a) Find H_2, H_3 and H_4. [2]

(b) Conjecture a formula for H_n in terms of n, for $n \in \mathbb{Z}^+$. [1]

(c) Prove by mathematical induction that your formula is valid for all $n \in \mathbb{Z}^+$. [5]

7. [Maximum mark: 9]

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f : x \to \begin{cases} -3x + 1 & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ 2x^2 - 3x + 1 & \text{for } x > 0 \end{cases}$.

By considering limits prove that f is

(a) continuous at $x = 0$; [4]

(b) differentiable at $x = 0$. [5]

8. [Maximum mark: 14]

The points A, B have coordinates $(-3, 0)$, $(5, 0)$ respectively. Consider the Apollonius circle C which is the locus of point P where

$$\frac{\mathrm{AP}}{\mathrm{BP}} = k \text{ for } k \neq 1.$$

Given that the centre of C has coordinates $(13, 0)$, find

(a) (i) the value of k;

(ii) the radius of C;

(iii) the x-intercepts of C. [11]

(b) Let M be any point on C and N be the x-intercept of C between A and B. Prove that $\mathrm{A\hat{M}N} = \mathrm{N\hat{M}B}$. [3]

Turn over

9. [Maximum mark: 11]

(a) Use the Euclidean algorithm to find $\gcd(162, 5982)$. [4]

(b) The relation R is defined on $\mathbb{Z}^+$ by nRm if and only if $\gcd(n, m) = 2$.

(i) By finding counterexamples show that R is neither reflexive nor transitive.

(ii) Write down the set of solutions of $nR6$. [7]

10. [Maximum mark: 10]

(a) Show that $2^n \equiv (-1)^n \pmod 3$, where $n \in \mathbb{N}$. [3]

(b) Hence show that an integer is divisible by 3 if and only if the difference between the sum of its binary (base 2) digits in even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3. [3]

(c) Express the hexadecimal (base 16) number ABBA_{16} in binary. [4]

11. [Maximum mark: 8]

The points P, Q and R, lie on the sides $[\text{AB}]$, $[\text{AC}]$ and $[\text{BC}]$, respectively, of the triangle ABC. The lines (AR), (BQ) and (CP) are concurrent.

Use Ceva's theorem to prove that $[\text{PQ}]$ is parallel to $[\text{BC}]$ if and only if R is the midpoint of $[\text{BC}]$.

12. [Maximum mark: 14]

In this question, x, y and z denote the coordinates of a point in three-dimensional Euclidean space with respect to fixed rectangular axes with origin O. The vector space of position vectors relative to O is denoted by $\mathbb{R}^3$.

(a) Explain why the set of position vectors of points whose coordinates satisfy $x - y - z = 1$ does not form a vector subspace of $\mathbb{R}^3$. [1]

(b) (i) Show that the set of position vectors of points whose coordinates satisfy $x - y - z = 0$ forms a vector subspace, V, of $\mathbb{R}^3$.

(ii) Determine an orthogonal basis for V of which one member is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(iii) Augment this basis with an orthogonal vector to form a basis for $\mathbb{R}^3$.

(iv) Express the position vector of the point with coordinates $(4, 0, -2)$ as a linear combination of these basis vectors. [13]

13. [Maximum mark: 11]

The discrete random variables X_n, $n \in \mathbb{Z}^+$ have probability generating functions given by $G_n(t) = \dfrac{t}{n}\left(\dfrac{t^n - 1}{t - 1}\right)$.

(a) Use the formula for the sum of a finite geometric series to show that

$$\mathrm{P}\left(X_n = k\right) = \begin{cases} \dfrac{1}{n} & \text{for } 1 \le k \le n \\ 0 & \text{otherwise} \end{cases}.$$ [4]

(b) Find $\mathrm{E}(X_n)$. [3]

Let X_{n-1} and X_{n+1} be independent.

(c) Find the set of values of n for which $\mathrm{E}\left(X_{n-1} \times X_{n+1}\right) < 2n$. [4]

14. [Maximum mark: 16]

A matrix $\boldsymbol{M}$ is called idempotent if $\boldsymbol{M}^2 = \boldsymbol{M}$.

(a) (i) Explain why $\boldsymbol{M}$ is a square matrix.

(ii) Find the set of possible values of $\det(\boldsymbol{M})$. [4]

The idempotent matrix $\boldsymbol{N}$ has the form

$$\boldsymbol{N} = \begin{pmatrix} a & -2a \\ a & -2a \end{pmatrix}$$

where $a \neq 0$.

(b) (i) Find the value of a.

(ii) Find the eigenvalues of $\boldsymbol{N}$.

(iii) Find corresponding eigenvectors. [12]